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Symmetric Minkowski planes ordered by separation

Helmut Karzel, Jarosław Kosiorek and Andrzej Matraś

Dedicated to Mario Marchi

Abstract. In Karzel et al. (J. Geom. 99:116–127, 2009) we introduced for a symmetric Minkowski plane $\mathfrak{M} := (P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$ an order concept by the notion of an orthogonal valuation for the circles of Λ and showed that there is a one to one correspondence between the valuations and the halforderings of the accompanying commutative field. Here we consider an order concept which is based on the notion of *separation* for quadruples of concyclic points and establish the connections between these two notions. Our main result (cf. Theorem 3.3) states that these concepts are equivalent.

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Introduction

Order concepts in geometry are usually based on both, *points* and *blocks*. Minkowski planes belong to the chain structures and so there appear two types of blocks, *generators* and *chains* latter also called *circles* (in the case of Minkowski planes). In [1] we showed that in a symmetric Minkowski plane $\mathfrak{M} := (P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$ an order structure can be introduced where only the set Λ of circles is involved, namely by a *valuation* which associates to each circle $C \in \Lambda$ a value $[C]$ which is 1 or -1. Then, since Λ can be endowed with a multiplication "$\cdot$" such that $(\Lambda, \cdot)$ becomes a group, it turns out that the valuation is a homomorphism of $(\Lambda, \cdot)$ in the cyclic group $(\{1, -1\}, \cdot)$ of order two hence $[A \cdot B] = [A] \cdot [B]$ for $A, B \in \Lambda$. Furthermore to a symmetric Minkowski plane $\mathfrak{M}$ there corresponds a commutative field $(\mathbf{F}, +, \cdot)$ such that $(\Lambda, \cdot)$ is isomorphic to the projective linear group $PGL(2, \mathbf{F})$ and if $[\,]$ is an orthogonal valuation of

$\mathfrak{M}$ then there is a halfordering η of the commutative field $(\mathbf{F}, +, \cdot)$ such that if $A_f \in GL(2, \mathbf{F})$ is a matrix representing a chain $A \in \Lambda$ then $[A] = \eta(det(A_f))$.

In [2] Kroll (cf. Subsection 1.6) defined order structures for Benz planes—and so also for Minkowski planes—starting inter alia from the notion of *separation*, i.e. a function which maps each quadruple (a, b, c, d) consisting of concyclic points with $a, b \neq c, d$ on a value $[a, b|c, d]$ in $\{1, -1\}$ such that certain conditions are satisfied (cf. Sect. 1.5).

In Sect. 1.5 we recall for symmetric Minkowski planes $\mathfrak{M}$ the notions *orthogonal valuation*, *order valuation* and then *halfordered* and *ordered* symmetric Minkowski planes $(\mathfrak{M}, [\ ])$ (based on valuations). Theorem 1.13. describes properties of these structures and Theorem 1.14. recalls relations to the halforderings and orderings of the corresponding commutative field $(\mathbf{F}, +, \cdot)$.

Following Kroll we introduce the concepts *separation* and *order separation* by conditions **(T1)**, **(T2)** and **(T1)**, **(T2)**, **(T3)**, respectively.

Theorem 2.1 shows how one can derive from an orthogonal valuation $[\ ]$ of a symmetric Minkowski plane $\mathfrak{M}$ a separation τ of $\mathfrak{M}$. The separation τ is harmonic iff the valuation $[\ ]$ is harmonic, and τ is an order separation iff $[\ ]$ is an order valuation.

Finally in Sect. 3 we start from a pair $(\mathfrak{M}, \tau)$ where $\mathfrak{M}$ is a symmetric Minkowski plane and τ a separation of $\mathfrak{M}$ and prove in Theorem 3.1 some properties of $(\mathfrak{M}, \tau)$. Then we show in Theorem 3.2 how one can derive from τ a halfordering η_τ for the corresponding field $(\mathbf{F}, +, \cdot)$. Combining these results starting from a halfordered symmetric Minkowski $(\mathfrak{M}, [\ ])$ we can derive firstly (as in [1]) a halfordering η of the corresponding field $(\mathbf{F}, +, \cdot)$ and secondly a separation τ of $\mathfrak{M}$ and then a halfordering η_τ of $(\mathbf{F}, +, \cdot)$. We show $\eta = \eta_\tau$ and this proves our main result Theorem 3.3.

1. Preliminaries and supplements

1.1. Hyperbola structures and Minkowski planes

In this paper we will use mostly the same notations as in [1]. Let $(P, \mathfrak{G}_1, \mathfrak{G}_2)$ be a net. As in [1] let $P^{(3)} := \{\{a, b, c\} \in \binom{P}{3} \mid \{a, b, c\}$ is joinable$\}$ and let $\mathfrak{C}$ be the set of all chains of $(P, \mathfrak{G}_1, \mathfrak{G}_2)$. The quadruple $(P, \mathfrak{C}, \mathfrak{G}_1, \mathfrak{G}_2)$ is then called *maximal chain structure*.

For $a, b, c, d \in P$ we add the notions $(a, b, c, d)^\square := \{ab, ba, cd, dc\}$ and $\{a, b, c, d\}^\square := \{\{a, b, c, d\}, (a, b, c, d)^\square, (a, c, d, b)^\square, (a, d, b, c)^\square\}$.
If $\{x, y, z, u\} \in \{a, b, c, d\}^\square$ then $\{x, y, z, u\}^\square = \{a, b, c, d\}^\square$.

A quadruple $(P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$ is called *hyperbola structure* if $\Lambda \subseteq \mathfrak{C}$ such that:

(I) $\forall \{a, b, c\} \in P^{(3)} \exists_1 L \in \Lambda : \ \{a, b, c\} \subseteq L-$ we set $(a, b, c)^\circ := L-$.

If $(P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$ is a hyperbola structure then a subset S of P is called *concyclic* if there is a *circle* $L \in \Lambda$ such that $S \subseteq L-$ if $|S| \geq 3$ then L is uniquely determined and we set then $S^\circ := L-$. A hyperbola structure $(P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$ is called *symmetric Minkowski plane* if the so called *"Symmetry Axiom"* is satisfied which we can express in the following form (cf. [1, (2.7)]) :

(S) If a quadruple (a, b, c, d) consists of concyclic points then also the set $(a, b, c, d)^\square$ is concyclic.

In a symmetric Minkowski plane, if $F := \{a, b, c, d\}$ is a set consisting of four distinct concyclic points then by Axiom **(S)**, the set $F^\square$ consists of four subsets of points and each subset consisting of four distinct concyclic points so that (by Axiom **(I)**) we can form the set

$$\Lambda_F := \{\{a, b, c, d\}^\circ, (a, b, c, d)^{\square\circ}, (a, c, d, b)^{\square\circ}, (a, d, b, c)^{\square\circ}\}$$

consisting of four distinct circles which are orthogonal in pairs. A quadruple (a, b, c, d) consisting of four distinct concyclic points is called *harmonic* if $\{c, d\} = (a, b, c)^\circ \cap (ab, ba, c)^\circ$. Let $P^{(4h)}$ denote the *set of all harmonic quadruples.*

From now on let $\mathfrak{M} = (P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$ be a symmetric Minkowski plane, let a circle $E \in \Lambda$ be fixed and for $A, B \in \mathfrak{C}$ let $A \cdot B := t(A, E, B) = \widetilde{AB}(E)$ then (cf. [1, p. 118 f]):

Theorem 1.1. *$(\mathfrak{C}, \cdot)$ is a group with the neutral element E, $t(A, B, C) = A \cdot B^{-1} \cdot C$ and the orthogonality of chains is described by:*

$$A \perp B \Leftrightarrow A \cdot B^{-1} = B \cdot A^{-1} \quad \textit{and} \quad A \neq B.$$

Moreover Λ is a subgroup of $(\mathfrak{C}, \cdot)$.

We recall that a quadruple (A, B, C, D) of circles is called a *quadrangle* if $D = t(A, B, C)$, a quadrangle is called a *square* if $A \perp B \perp C \perp D \perp A$ and a *total square* if moreover $A \perp C$ and $B \perp D$. If F is a set of four distinct concyclic points then Λ_F is a total square,

1.2. Properties of symmetric Minkowski planes

In a symmetric Minkowski planes $\mathfrak{M} = (P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$ we define $\Gamma^- := \{\widetilde{AB} \mid A, B \in \Lambda\}$, denote by $\Gamma := \langle \Gamma^- \rangle$ the group generated by Γ^- and set

$$(\Lambda \times \Lambda)_\perp := \{(A, B) \in (\Lambda \times \Lambda) \mid A \perp B\} \text{ and}$$
$$(\Lambda \times \Lambda)_{\perp,s} := \{(A, B) \in (\Lambda \times \Lambda)_\perp \mid A \cap B \neq \emptyset\}.$$

For a set S of concyclic points with $|S| \geq 3$ there is exactly one circle in Λ containing S which shall be denoted also by S°.

Finally the notions of *1- and 2-perspectivities* can be defined:

Let $A, B \in \Lambda$ then the map

$$[A \xrightarrow{1} B] : A \to B; x \mapsto xB \text{ and } [A \xrightarrow{2} B] : A \to B; x \mapsto Bx$$

is called *1-* and *2-perspectivity*, respectively.

From [1, Section 2.1, (2.5),(2.6),(2.7)] follows:

Theorem 1.2. *Let $A, B, C, X \in \Lambda$ and let $(a, b, c, d) \in X$ with $a, b \neq c, d$, then:*

(1) $\Gamma \leqslant Aut(P, \Lambda, \mathfrak{G}_1 \cup \mathfrak{G}_2)$.
(2) *If $a \neq b$ then $\{ab, ba, cd\} \in P^{(3)}$, $dc \in K := (ab, ba, cd)^\circ$ and $K \perp X$.*
(3) *If $|\{a, b, c, d\}| \geqq 3$ then $X = (a, b, c, d)^\circ$ and $K := ((a, b, c, d)^\square)^\circ$ are uniquely determined circles of Λ and $X \perp K$; if $a = b$ or $c = d$ then $K = ((c, d, a)^\beta)^\circ$ and $a \in X \cap K$ or $K = ((a, b, c)^\beta)^\circ$ and $c \in X \cap K$, respectively.*
(4) *If $|\{a, b, c, d\}| = 4$ and $F := \{a, b, c, d\} \subseteq E$ then Λ_F is a subgroup of $(\Lambda, \cdot)$ and consists of four circles which are orthogonal in pairs (cf. Theorem 1.1), i.e. Λ_F is a total square.*
(5) *If $A, B \perp C$, $A \cap C = B \cap C = \{p, q\}$ and $p \neq q$ then $A = B$.*

Proof. (5) Assume $A \neq B$ hence $A \cap B = \{p, q\}$ then by [1, (3.5.1)], $pq, qp \in C$. But since $p, q \in C$ and $p \neq q$ we have $pq, qp \notin C$. Consequently $A = B$. □

Theorem 1.3. *Let $K \in \Lambda$, $a, b, c, d, e \in K$ with $a, b \neq c, d, e$ and $a \neq b$ and let $L_c := \{ab, ba, de\}^\circ$, $L_d := \{ab, ba, ec\}^\circ$, $L_e := \{ab, ba, cd\}^\circ$ and $M := \widetilde{L_e L_c}(L_d) = L_e \cdot L_d^{-1} \cdot L_c$ then $M = \{ab, ba, d\}^\circ$, $M \perp K$ and $d \in M \cap K$, i.e. $(M, K) \in (\Lambda \times \Lambda)_{\perp s}$.*

Proof. Since $\{ab, ba, cd, dc\} \subseteq L_e$, $\{ab, ba, ce, ec\} \subseteq L_d$ and $\{ab, ba, de, ed\} \subseteq L_c$ we have $ab, ba \in Fix\widetilde{L_e L_c}$ and $\widetilde{L_e L_c}(ce) = (de)(cd) = d$ hence $M = \widetilde{L_e L_c}(L_d) = \{ab, ba, d\}^\circ$ with $M \perp K$ and $d \in M \cap K$ and so $(M, K) \in (\Lambda \times \Lambda)_{\perp s}$. □

By [1, (2.5.2) and (3.9)] we have:

Theorem 1.4. *The set $\Gamma^- := \{\widetilde{AB} \mid A, B \in \Lambda\}$ consists of automorphisms of (P, Λ) and acts transitively on the set of circles Λ preserving orthogonality and if γ is an element of the group Γ generated by Γ^- then there exist exactly two elements $A, B \in \Lambda$ such that either $\gamma = \widetilde{AB}$ or $\gamma = \widetilde{AB} \circ \widetilde{E}$. Moreover $\widetilde{AB} \circ \widetilde{CD} = \widetilde{(A \cdot D^{-1})(C^{-1} \cdot B)} \circ \widetilde{E}$ and $\Gamma^+ := \widetilde{E} \circ \Gamma^-$ acts regularly on the set $P^{(3)}$.*

Therefore, if $A, B \in \Lambda$ and $C := \widetilde{AE}(B) = A \cdot B^{-1} \cdot E = A \cdot B^{-1}$ we have:

$$(A, B) \in (\Lambda \times \Lambda)_{\perp,s} \Leftrightarrow (E, C) \in (\Lambda \times \Lambda)_{\perp,s}.$$

Consequently, $\forall (A, B), (C, D) \in (\Lambda \times \Lambda)_{\perp,s} :\ |A \cap B| = |C \cap D|$.

1.3. The corresponding field and the characteristic of a symmetric Minkowski plane

As in [1] we denote by $(\mathbf{F}, +, \cdot)$ the commutative field corresponding to the symmetric Minkowski plane $\mathfrak{M}$ (cf. [1, sec.1]) and we define $char\mathfrak{M} := char\mathbf{F}$.

If $\binom{c_1, c_2}{c_3, c_4} \in GL(2, \mathbf{F})$ represents the circle C then $C \perp E$ implies $c_1 = -c_4$ and if $c_1 = -c_4$ then:

$C \cap E \neq \emptyset \Leftrightarrow$ the characteristic equation $\Xi_C(t) = t^2 - (c_1^2 + c_2 \cdot c_3) = 0$ is solvable $\Leftrightarrow (c_1^2 + c_2 \cdot c_3) \in \mathbf{F}^{(2)} := \{\lambda^2 \mid \lambda \in \mathbf{F} \setminus \{0\}\}$.

Hence if $(c_1^2 + c_2 \cdot c_3) \in \mathbf{F}^{(2)}$ then: $|C \cap E| = 2$ if $char\mathbf{F} \neq 2$ and $|C \cap E| = 1$ if $char\mathbf{F} = 2$. This supplements the statements of [1, (3.7)]:

Theorem 1.5. *Let $(A, B) \in (\Lambda \times \Lambda)_{\perp,s}$ then:*

(1) $|A \cap B| = 2 \Leftrightarrow char\mathfrak{M} \neq 2$.
(2) $|A \cap B| = 1 \Leftrightarrow char\mathfrak{M} = 2$.
(3) *If $char\mathfrak{M} \neq 2$ then $\forall (a, b, c) \in P^{(3)}\ \exists_1 d \in P$ such that (a, b, c, d) is a harmonic quadruple and d is given by $\{a, b, c\}^\circ \cap \{ab, ba, c\}^\circ = \{c, d\}$. If $char\mathfrak{M} = 2$, there are no harmonic quadruples.*

Now let $C, D \in \Lambda$ be circles with $C, D \perp E$ and $C \cap E \neq \emptyset$, $D \cap E \neq \emptyset$ and let $\left(\begin{smallmatrix} c_1, c_2 \\ c_3, -c_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} d_1, d_2 \\ d_3, -d_1 \end{smallmatrix}\right) \in GL(2, \mathbf{F})$ be a representing matrices. Then we may assume $d_1^2 + d_2 \cdot d_3 = c_1^2 + c_2 \cdot c_3 = 1$ and moreover $c_1 = 1$ if $c_2 \cdot c_3 = 0$ and $d_1 = 1$ if $d_2 \cdot d_3 = 0$. Let $M_C := \left(\begin{smallmatrix} 0, & c_2 \\ 1, -c_1 \end{smallmatrix}\right)$ if $c_2 \neq 0$, $M_C := \left(\begin{smallmatrix} c_1, 1 \\ c_3, 0 \end{smallmatrix}\right)$ if $c_2 = 0 \neq c_3$ and $M_C := \left(\begin{smallmatrix} 1, & 1 \\ 1, -1 \end{smallmatrix}\right)$ if $c_2 = c_3 = 0$ and $char\mathbf{F} \neq 2$ and let M_D be defined accordingly. Then if $L \in \Lambda$ is the circle represented by the matrix $M_D \cdot M_C^{-1}$. Then the map $\widetilde{LL^{-1}}$ fixes E and maps C onto D. This gives us:

Theorem 1.6. Γ *acts transitively on the set $(\Lambda \times \Lambda)_{\perp s}$.*

In the same way using representing matrices one confirms the *Three Reflection Theorems*:

Theorem 1.7. *Let $A, B, C \in \Lambda$ with $A \cap B \cap C \neq \emptyset$ and $p \in A \cap B \cap C$ then:*

(1) *If $|A \cap B \cap C| \geq 2$ then: $\exists D \in \Lambda : \widetilde{A} \circ \widetilde{B} \circ \widetilde{C} = \widetilde{D}$.*
(2) *If there is an $L \in \Lambda$ with $p \in L$ and $A, B, C \perp L$ then: $\exists\ D \in \Lambda$: $\widetilde{A} \circ \widetilde{B} \circ \widetilde{C} = \widetilde{D}$ (and then $p \in D$ and $D \perp L$ or $D = L$).*

By the definition of the maps $\widetilde{AB}$ one obtains the following representations of 1- and 2-perspectivities:

Proposition 1.8. *Let $A, B, C \in \Lambda$ and $D := \widetilde{AB}(C)$ then:*

(1) $[A \xrightarrow{1} B] = \widetilde{BA}_{|A}$ *and* $[A \xrightarrow{2} B] = \widetilde{AB}_{|A}$.
(2) $\widetilde{AB}_{|C} = [A \xrightarrow{2} D] \circ [C \xrightarrow{1} A] = [B \xrightarrow{1} D] \circ [C \xrightarrow{2} B]$.
(3) $\bar{A}\ :\ E \to E; x \mapsto E(xA) = [A \xrightarrow{2} E] \circ [E \xrightarrow{1} A](x) = \widetilde{AE} \circ \widetilde{AE}(x)$.
(4) *If $A \perp B$ then $\psi := [B \xrightarrow{2} A] \circ [A \xrightarrow{1} B] = \widetilde{BA} \circ \widetilde{BA}_{|A}$ and if for $x \in A$, $x' := \widetilde{B}(x)$ then:*

$$x' = \psi(x) \in A, (x')' = x, xx' \in B \quad \text{and} \quad \widetilde{A}(xx') = x'x \in B.$$

Theorem 1.9. *Let (a, b, c, d) and (a', b', c', d') be harmonic quadruples and let $\gamma \in \Gamma$ then:*

(1) $(b,a,c,d),(c,d,a,b),(ab,ba,cd,dc),(ab,ba,c,d)$ *and* $(\gamma(a),\gamma(b),\gamma(c),\gamma(d))$ *are harmonic quadruples.*

(2) $\exists\, \sigma \in \Gamma:\ \sigma(a)=a',\, \sigma(b)=b',\, \sigma(c)=c',\, \sigma(d)=d'$.

(3) *The circles* $K := \{a,b,c\}^\circ$, $L := \{ab,ba,cd\}^\circ$, $M := \{ab,ba,c\}^\circ$ *and* $N := \{a,b,cd\}^\circ$ *are orthogonal in pairs hence* $\mathfrak{H} := (K,L,M,N)$ *is a total square in* Λ *and* $(K,M),(K,N),(L,M),(L,N) \in (\Lambda\times\Lambda)_{\perp,s}$ *and* $K\cap M=\{c,d\}$, $K\cap N=\{a,b\}$, $L\cap M=\{ab,ba\}$ *and* $L\cap N=\{cd,dc\}$. $K^{-1}\cdot\{K,L,M,N\}$ *is a subgroup of* $(\Lambda,\cdot)$ *isomorphic to the Klein four group.*

Proof. (1) The definition *harmonic* is symmetric in the first two and in the last two arguments and from $\{c,d\}=\{a,b,c\}^\circ\cap\{ab,ba,c\}^\circ$ we obtain $\{a,b,c\}^\circ = (\{c,d,a\}^\circ$ and ab,ba,c,d are four distinct concyclic points hence by **(S)**, $(ab,ba,c,d)^\square = \{a,b,cd,dc\}$ is concyclic, i.e. $\{cd,dc,a\}^\circ \ni a,b$ and so $\{a,b\} = \{c,d,a\}^\circ\cap\{cd,dc,a\}^\circ$, i.e. $(c,d,a,b)\in P^{(4h)}$.

(2) By Theorem 1.6 there is exactly one $\sigma\in\Gamma^+$ with $\sigma(a)=a'$, $\sigma(b)=b'$, $\sigma(c)=c'$ and so $\{\sigma(c),\sigma(d)\}=\{\sigma(a),\sigma(b),\sigma(c)\}^\circ\cap\{\sigma(a)\sigma(b),\sigma(b)\sigma(a),\sigma(c)\}^\circ=\{a',b',c'\}^\circ\cap\{a'b',b'a',c'\}^\circ=\{c',d'\}$ implying $\sigma(d)=d'$. □

Theorem 1.10. *Let* a,b,c,d *be four distinct concyclic points,* $K:=\{a,b,c\}^\circ$, $C:=\{bc,cb,a\}^\circ, D:=\{bd,db,a\}^\circ, H:=\{cd,dc,a\}^\circ$ *and let* $B\in\Lambda$ [*according to Theorem* 1.7(2)] *be such that* $\widetilde{B}=\widetilde{D}\circ\widetilde{H}\circ\widetilde{C}$. *Then* $\widetilde{B}(\widetilde{C}(d))=\widetilde{D}(c)$ *and* $(a,b,\widetilde{C}(d),\widetilde{D}(c))$ *is harmonic.*

Proof. By the definitions we have: $a\in C,H,D,K;\ \ C,H,D\perp K$, $\widetilde{C}(b)=c$, $\widetilde{H}(c)=d$ and $\widetilde{D}(d)=b$ hence, by Theorem 1.9(2), $B\perp K$, $\widetilde{B}(a)=\widetilde{D}\circ\widetilde{H}\circ\widetilde{C}(a)=a$, $\widetilde{B}(b)=\widetilde{D}\circ\widetilde{H}\circ\widetilde{C}(b)=b$ and $\widetilde{B}(\widetilde{C}(d))=\widetilde{D}\circ\widetilde{H}\circ\widetilde{C}(\widetilde{C}(d))=\widetilde{D}\circ\widetilde{H}(d)=\widetilde{D}(c)$. Since moreover $\widetilde{C}(d),\widetilde{D}(c)\in K$ we have $\{\widetilde{C}(d),\widetilde{D}(c)\}=\{a,b,\widetilde{C}(d)\}^\circ\cap\{ab,ba,\widetilde{C}(d)\}^\circ$ i.e. $(a,b,\widetilde{C}(d),\widetilde{D}(c))$ is harmonic. □

1.4. Orthogonal valuations and separations in symmetric Minkowski planes

In [1, p. 120 and p. 126] a map $[\ ]:\Lambda\to\{1,-1\}; X\mapsto[X]$ was called *orthogonal valuation* if

(O 1) For each square $(A,B,C,D)\ :\ [A]\cdot[B]\cdot[C]\cdot[D]=1.$ and *homomorphic valuation* if this equation holds true even for each quadrangle. An orthogonal valuation is called *order valuation* if

(O 2) For each total square (A,B,C,D) : exactly two of the values $[A],[B]$, $[C],[D]$ equal 1.

By Theorems 1.4, 1.5 and [1, (4.5) and (4.8)] we have:

Theorem 1.11. *Let* $[\ ]:\Lambda\to\{1,-1\}; X\mapsto[X]$ *be an orthogonal valuation of* $\mathfrak{M}$, *let* $E\in\Lambda$ *with* $[E]=1$ *be fixed, for* $A,B\in\Lambda$ *let* $A\cdot B:=t(A,E,B)$ *and for* $\gamma=\widetilde{AB}\in\Gamma^-$ *or* $\gamma=\widetilde{AB}\circ\widetilde{E}\in\Gamma^+$ *let* $[\gamma]':=[A]\cdot[B]$. *Then:*

(1) $[\ ]$ *is a homomorphic valuation hence* $\forall A, B, C \in \Lambda : [t(A, B, C)] = [A \cdot B^{-1} \cdot C] = [A] \cdot [B] \cdot [C]$ *and* $[A \cdot B] = [A] \cdot [B]$.
(2) $[\ ]' : (\Gamma, \circ) \to (\{1, -1\}, \cdot); \gamma \mapsto [\gamma]'$ *is a homomorphism.*
(3) $\forall X \in \Lambda, \forall \gamma \in \Gamma : [\gamma(X)] = [\gamma]' \cdot [X]$.
(4) *There is an* $\iota \in \{1, -1\}$ *such that:* $\forall (A, B) \in (\Lambda \times \Lambda)_{\perp s} :\ [A] \cdot [B] = \iota$.
(5) *If* $[\ ]$ *is an order valuation then* $\iota = -1$.

An orthogonal valuation $[\ ]$ is called *harmonic* if $\iota = -1$ and *anharmonic* if $\iota = 1$.

Let $P^{(4)} := \{(a, b, c, d) \in P^4 \mid \{a, b, c, d\} \text{ concyclic} \wedge a, b \neq c, d\}$.

A map $\tau : P^{(4)} \to (\{1, -1\}, \cdot)$; $(a, b, c, d) \mapsto [a, b|c, d]$ is called a *separation* if the following conditions are satisfied:

(T1) For all concyclic $a, b, c, d, e \in P$ with $a, b \neq c, d, e$ holds:
$[a, b|c, d] \cdot [a, b|d, e] = [a, b|c, e]$.

(T2) For all $(a, b, c, d) \in P^{(4)}$, for all $L \in \Lambda$:
$[a, b|c, d] = [La, Lb|Lc, Ld] = [aL, bL|cL, dL]$, i.e. the separation function is invariant by 1- and 2-perspectivities.

If moreover the axiom

(T3) $\forall (a, b, c, d) \in P^{(4)}$ with $a \neq b, c \neq d$: exactly one of the values $[a, b|c, d]$, $[a, c|d, b]$, $[a, d|b, c]$ is -1

is valid then τ is called an *order separation*.

A separation is called *harmonic* if for a harmonic quadruple (a, b, c, d) (and then for all) the value $[a, b|c, d]$ is -1. This definition corresponds E. Sperner's definition of a harmonic separation.

1.5. Kroll's order concepts for Minkowski planes

Kroll [2] developed a thoroughly theory of order questions for Benz planes. Benz planes contain as a subclass the Minkowski planes. We present some of Kroll's concepts specialized for Minkowski planes $\mathfrak{M} := (P, \Lambda, \mathfrak{G}_1, \mathfrak{G}_2)$: Let $(\Lambda \times P \times P)^* := \{(A, b, c) \in \Lambda \times P \times P \mid b, c \notin A\}$. Then a map $\alpha : (\Lambda \times P \times P)^* \to \{1, -1\}; (A, b, c) \mapsto (A|b, c)$ is called *order function* of $\mathfrak{M}$ if :

O 1 $\forall A \in \Lambda$ and $\forall x, y, z \in P \setminus A : (A|x, y) \cdot (A|y, z) \cdot (A|z, x) = 1$.
O 2 (circle relation) $\forall A, B, C \in \Lambda$ such that $A \cap C = B \cap C$ and $|A \cap C| \geq 1$ and $\forall x, y \in C \setminus A : (A|x, y) = (B|x, y,)$.
OV $\forall G \in \mathfrak{G}_1 \cup \mathfrak{G}_2,\ \forall A, B \in \Lambda$ with $A \cap G = B \cap G$ and $\forall c, d \in G \setminus A$: $(A|c, d) = (B|c, d)$ (cf. [2, p. 224 and 231]).

If $(\mathfrak{M}, \alpha)$ is a Minkowski plane with an order function, Kroll derives a separation τ_α by:

(V) For $K \in \Lambda$, for $a, b, c, d \in K$ with $a, b \neq c, d$ and $A, L \in \Lambda$ with $A \cap K = \{a\}, L \cap K = \{a, b\}$ let $[a, b|c, d] := (A|c, d) \cdot (L|c, d)$. ([2, p. 234])

He shows that τ_α satisfies the conditions **(T1)** and **(T2)** if $\mathfrak{M}$ is a symmetric Minkowski plane (cf. [2, p. 234 and p. 237], $\mathbf{T}4^*i$). Conversely every separation of a symmetric Minkowski plane is a separation in the sense of Kroll.

2. From an orthogonal valuation to a separation

Theorem 2.1. *Let $(\mathfrak{M}, [\ ])$ be a halfordered symmetric Minkowski plane, let ι be the value corresponding to $(\mathfrak{M}, [\ ])$ according to Theorem 1.11(4) and let $\tau : P^{(4)} \to \{1, -1\}; (a, b, c, d) \mapsto [a, b|c, d]$ be the map defined by $[a, b|c, d] := 1$ if $a = b$ and $[a, b|c, d] := \iota \cdot [\{a, b, c, d\}^\circ] \cdot [(a, b, c, d)^{\square\circ}]$ if $a \neq b$ then:*

(1) *τ is a separation and if (a, b, c, d) is harmonic then $[a, b|c, d] = \iota$. Hence the derived separation τ is harmonic if and only if the valuation $[\ ]$ is harmonic.*
(2) *If $[\ ]$ is an order valuation then τ is an order separation.*

Proof. **(T1)** Let $(a, b, c, d), (a, b, d, e) \in P^{(4)}$. If $a = b$ then $[a, b|c, d] = [a, b|d, e] = [a, b|c, e] = 1$ and the equation **(T1)** is valid. Let $a \neq b$ then $(a, b, c), (a, b, d) \in P^{(3)}$, so we can form the circles $K := \{a, b, c\}^\circ$ and $K' := \{a, b, d\}^\circ$ and by the assumption, $(a, b, c, d), (a, b, d, e) \in P^{(4)}$, we have $a, b, c, d, e \in K = K'$. If we define the circles L_c, L_d, L_e and M according to Theorem 1.3 then $(M, K) \in (\Lambda \times \Lambda)_{\perp s}$ hence by Theorem 1.11(4), $[M] \cdot [K] = \iota$ and by definition of τ we have $[a, b|c, d] := \iota \cdot [K] \cdot [L_e]$, $[a, b|d, e] := \iota \cdot [K] \cdot [L_c]$ and $[a, b|e, c] := \iota \cdot [K] \cdot [L_d]$. By Theorem 1.11(1)), this implies:

$[a, b|c, d] \cdot [a, b|d, e] \cdot [a, b|e, c] = \iota \cdot [K] \cdot [L_e] \cdot \iota \cdot [K] \cdot [L_c] \cdot \iota \cdot [K] \cdot [L_d] = \iota \cdot [K] \cdot [L_e \cdot L_d^{-1} \cdot L_c] = \iota \cdot [K] \cdot [M] = \iota \cdot \iota = 1.$

(T2) Let $K, L \in \Lambda, \{a, b, c, d\} \in \binom{K}{4}$ and $H := \{ab, ba, cd\}^\circ$ then by Theorem 1.2(2), $H \perp K$ hence $[a, b|c, d] = [K] \cdot [H] \cdot \iota$ and by Proposition 1.8,

$$\pi_1 := [K \xrightarrow{1} L] = \widetilde{LK}_{|K}, \quad \pi_2 := [K \xrightarrow{2} L] = \widetilde{KL}_{|K}.$$

Then $\widetilde{KL}(K) = L = \widetilde{LK}(K)$ and if $H_1 := \widetilde{LK}(H)$, $H_2 := \widetilde{KL}(H)$ then $H_1, H_2 \perp L$ and so $H_i = ((\pi_i(a), \pi_i(b), \pi_i(c), \pi_i(d))^\square)^\circ$. Thus

$[\pi_i(a), \pi_i(b)|\pi_i(c), \pi_i(d)] = [L] \cdot [H_i] \cdot \iota$ with $H_1 = L \cdot H^{-1} \cdot K, H_2 = K \cdot H^{-1} \cdot L$ implying by Theorem 1.11(1), $[H_i] = [K] \cdot [H] \cdot [L]$. Therefore $\iota \cdot [\pi_i(a), \pi_i(b)|\pi_i(c), \pi_i(d)] = [K] \cdot [H] \cdot [L] \cdot [L] = [K] \cdot [H] = \iota \cdot [a, b|c, d]$.

Now let (a, b, c, d) be harmonic and let K, L, M, N be defined according to Theorem 1.9(3). Then by Theorem 1.9(3) and Theorem 1.11(4), $[a, b|c, d] = [K] \cdot [L] \cdot \iota = [K] \cdot [M] \cdot [M] \cdot [L] \cdot \iota = \iota \cdot \iota \cdot \iota = \iota$.

(2) By Theorem 1.11(5), $\iota = -1$. Therefore using the notations of Theorem 1.2(3) we have:

$$[a, b|c, d] = -[X] \cdot [Y], \quad [a, c|d, b] = -[X] \cdot [Z], \quad [a, d|b, c] = -[X] \cdot [U],$$

by Theorem 1.2(4) the quadruple (X, Y, Z, U) is a total square and so by **(O2)**, exactly two of the values $[X], [Y], [Z], [U]$ are 1. But from this observations follows **(T3)**. $\square$

3. From a separation to a halfordering of the corresponding field

We start from the assumption that the Minkowski plane $\mathfrak{M}$ is provided with a separation τ.

Theorem 3.1. *Let $(a, b, c, d) \in P^{(4)}$ with $a \neq b; c \neq d$ and let $\gamma \in \Gamma$ then:*

1. $[\gamma(a), \gamma(b)|\gamma(c), \gamma(d)] = [a, b|c, d]$.
2. $[a, b|c, d] \cdot [a, c|d, b] \cdot [a, d|b, c] = [a', b'|c', d']$ *where (a', b', c', d') is any harmonic point quadruple.*

Proof. (1) Follows from Proposition 1.8 and **(T2)**.

(2) Let $C := \{bc, cb, a\}^\circ$ and $D := \{bd, db, a\}^\circ$. Then $\widetilde{C}(a) = a$, $\widetilde{C}(b) = c$, $\widetilde{C}(c) = b$ and $\widetilde{D}(a) = a$, $\widetilde{D}(b) = d$ and so by (1), $[a, c|d, b] = [a, b|\widetilde{C}(d), c]$ and $[a, d|b, c] = [a, b|d, \widetilde{D}(c)]$. By (T1) we have $[a, b|c, d] \cdot [a, c|d, b] \cdot [a, d|b, c] = [a, b|c, d] \cdot [a, b|\widetilde{C}(d), c] \cdot [a, b|d, \widetilde{D}(c)] = [a, b|d, \widetilde{C}(d)] \cdot [a, b|d, \widetilde{D}(c)] = [a, b|\widetilde{C}(d), \widetilde{D}(c)]$. By Theorem 1.10, $[a, b|\widetilde{C}(d), \widetilde{D}(c)]$ is a harmonic quadruple. By Theorem 1.9 and **(T2)** the τ-value for all harmonic quadruples is the same. $\square$

We recall that the commutative field $(\mathbf{F}, +, \cdot)$ corresponding to the symmetric Minkowski plane $\mathfrak{M}$ can be obtained as follows:

Theorem 3.2. *Let $0, 1, \infty$ be three distinct points of a fixed circle $E \in \Lambda$ and let $\mathbf{F} := E \setminus \{\infty\}$. For $a, b \in \mathbf{F}$ and $c \in \mathbf{F}^* := \mathbf{F} \setminus \{0\}$ let A^+, $C^\cdot \in \Lambda$ be circles determined by: $A^+ := E$ if $a = 0$, $0a \in A^+$ and $A^+ \cap E = \{\infty\}$ if $a \neq 0$, $C^\cdot := \{0, \infty, 1c\}^\circ$ and let $\overline{A^+} := \widetilde{A^+E} \circ \widetilde{A^+E}_{|E}$, $\overline{C^\cdot} := \widetilde{C^\cdot E} \circ \widetilde{C^\cdot E}_{|E}$ [cf. Proposition 1.8(3)], $a + b := \overline{A^+}(b)$, $c \cdot b := \overline{C^\cdot}(b)$. Then:*

(1) $(\mathbf{F}, +, \cdot)$ *is a commutative field.*
(2) *The function $\eta : \mathbf{F}^* \to \{1, -1\}$; $x \mapsto [0, \infty|1, x]$ induced by the separation τ is a halfordering of $(\mathbf{F}, +, \cdot)$.*
(3) *If τ is an order separation then η is an ordering of $(\mathbf{F}, +, \cdot)$.*

Proof. (2) Let $a, b \in \mathbf{F}^*$. By the definition of $\overline{A^\cdot}$, $\overline{A^\cdot}(0) = 0$, $\overline{A^+}(\infty) = \infty$, $\overline{A^\cdot}(1) = a$ and $\overline{A^\cdot}(b) = a \cdot b$ and so by Theorem 3.1, $[0, \infty|1, b] = [0, \infty|a, a \cdot b]$ hence together with **(T1)**, $\eta(a \cdot b) = [0, \infty|1, a \cdot b] = [0, \infty|1, a] \cdot [0, \infty|a, a \cdot b] = [0, \infty|1, a] \cdot [0, \infty|1, b] = \eta(a) \cdot \eta(b)$.

Since $\widetilde{A^+E}(\infty) = \infty$ we have by Theorem 3.1, $[\infty, x|y, z] = [\infty, a+x|a+y, a+z]$ for $x, y, z \in \mathbf{F}$. Now let $a \neq 0, 1$. Then by using this formula and **(T2)**,

$$[\infty, 0|1, a] = \eta(a),$$

$$[\infty, 1|a, 0] = [\infty, 0|a-1, -1] = [\infty, 0|1, -1] \cdot [\infty, 0|1, a-1]$$

$$= \eta(-1) \cdot \eta(a-1) = \eta(1-a) \text{ and}$$

$$[\infty, a|0, 1] = [\infty, 0|-a, 1-a] = [\infty, 0|1, -a] \cdot [\infty, 0|1, 1-a]$$

$$= \eta(-a) \cdot \eta(1-a) = \eta(a^2 - a) = \eta(a) \cdot \eta(a-1).$$

(3) If τ is an order separation then exactly one of these three values is equal -1. Hence if $\eta(a) = 1$ then $-\eta(1-a) = \eta(a-1) = \eta((-1) \cdot ((1-a)) = \eta(-1) \cdot \eta(1-a)$ thus $\eta(-1) = -1$ and so $\eta(-a) = \eta((-1) \cdot a) = -\eta(a)$. Now let $x, y \in \mathbf{F}^*$ with $\eta(x) = \eta(y) = 1$. For $a := -x$ we obtain $\eta(-x) = -1$ and so $\eta(1-a) = \eta(1+x) = 1$. By $\eta(x^{-1} \cdot y) = (\eta(x))^{-1} \cdot \eta(y) = 1$, this implies $\eta(x+y) = \eta(x \cdot (1 + x^{-1}y) = \eta(x) \cdot \eta(1 + x^{-1}y) = 1$ and this is the monotony law for the addition. $\square$

Starting from a symmetric Minkowski plane $\mathfrak{M}$ endowed with an orthogonal valuation $[\,]$, we derived a separtion τ for $\mathfrak{M}$ via Theorem 2.1 and from the separation τ, a halfordering η_1 for the corresponding field $(\mathbf{F}, +, \cdot)$ via Theorem 3.2. We show that η_1 and the halfordering η_2 derived from $(\mathfrak{M}, [\,])$ via [1](4.7) are equal. Let $x \in \mathbf{F}^*$ and $X := \{0\infty, \infty 0, 1x\}^\circ$. Then $X \perp E$, by Theorem 1.11(1), $[E] = 1$ and so $\eta_1(x) = [0, \infty|1, x] = [E] \cdot [X] \cdot \iota = [X] \cdot \iota$ and moreover, $\left(\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right)$ is a matrix representing X. Therefore by [1](4.7), $[X] = \eta_2(det(\left(\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right)) = \eta_2(-x) = \eta_2(-1) \cdot \eta_2(x)$ hence $\eta_1(x) = \iota \cdot \eta_2(-1) \cdot \eta_2(x)$. The circles E and I represented by the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (with the determinants 1 and -1) are orthogonal and containing the point 1. Hence $(E, I) \in (\Lambda \times \Lambda)_{\perp s}$ and so by Theorem 1.11(4), $\iota = [E] \cdot [I] = \eta_2(1) \cdot \eta_2(-1) = \eta_2(-1)$ and so $\eta_1(x) = \eta_2(x)$ for all $x \in \mathbf{F}^*$. This gives us our main result:

Theorem 3.3. *Let $\mathfrak{M}$ be a symmetric Minkowski plane and let $(\mathbf{F}, +, \cdot)$ be the corresponding commutative field. Then there are a one to one correspondences between the orthogonal valuations of $\mathfrak{M}$, the separations of $\mathfrak{M}$ and the halforderings of $(\mathbf{F}, +, \cdot)$ where the order valuations correspond with the order separations and with the orderings of the field.*

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Helmut Karzel
Zentrum Mathematik
T.U. München
80290 Munich
Germany
e-mail: karzel@ma.tum.de

Jarosław Kosiorek and Andrzej Matraś
Faculty of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn
Żolnierska 14
10-561 Olsztyn
Poland
e-mail: kosiorek@matman.uwm.edu.pl
e-mail: matras@uwm.edu.pl

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